Weak Pion Production*

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The matrix element for the interaction $W+N \to \pi+N$ is studied, where W is a virtual intermediate boson for the weak interactions (or just the weak current). Weak pion production—production of a pion by high-energy neutrino collisions with nucleons—is governed by this matrix element. The main case of interest is in the energy region where the pion-nucleon 3-3 resonance is dominant. Formulas are derived for solving the problem in this region.

I. INTRODUCTION

IT is theoretically possible to study weak interactions at high energies by means of reactions induced by a neutrino beam obtained from the decay of pions and kaons in flight. The practical possibilities of such neutrino beams are now being investigated; theoretical work on neutrino interactions has already been done by various authors.[1] The most feasible experiment is the charge-exchange scattering of neutrinos (or antineutrinos) and nucleons.[1] As the neutrino energy under consideration for experiment at present and in the near future is in the high-energy region (≤ 5 BeV, say), it is of interest also to consider collisions in which a pion is produced. Here such a process is called weak pion production.

This process is very similar to the electroproduction of pions off nucleons. This paper is written in the same spirit as the electroproduction calculations given elsewhere.[2] In particular, in the region of the pion-nucleon final state in which the 3-3 isobar is expected to dominate, formulas are obtained to solve the problem.

II. WEAK INTERACTION THEORY

The interaction Lagrangian for the simplest possible weak interaction, muon decay, is

$$\mathcal{L}_{\text{int}}=(G/\sqrt{2})[\bar{\nu}'\gamma_\alpha(1+\gamma_5)\mu][\bar{\nu}\gamma_\alpha(1+\gamma_5)e]^\dagger+\text{H.c.} \quad (1)$$

(ν is written for the neutrino associated with that of the electron; ν' that associated with the muon.) If we consider weak interactions in which strongly interacting particles are involved, but only allow strangeness preserving processes we can write, assuming a "universal" weak interaction

$$\mathcal{L}_{\text{int}}=(G/\sqrt{2})[V_\alpha+P_\alpha] \times[\bar{\nu}\gamma_\alpha(1+\gamma_5)e+\bar{\nu}'\gamma_\alpha(1+\gamma_5)\mu]^\dagger+\text{H.c.}, \quad (2)$$

where V_α and P_α are the vector and axial vector weak currents of the strongly interacting particles. From beta-decay experiments we expect that $G(V_\alpha+P_\alpha)$ is like its analog in the leptonic case. In the limit of zero momentum transfer,

$$G\langle p|V_\alpha|n\rangle \to G_V\bar{\psi}_p\gamma_\alpha\tau_+\psi_n, \quad (3a)$$

$$G\langle p|P_\alpha|n\rangle \to -G_A\bar{\psi}_p\gamma_\alpha\gamma_5\tau_+\psi_n. \quad (3b)$$

G_V and G_A are the usual Fermi and Gamow-Teller coupling constants of nuclear theory.

Experimentally it turns out that $G_V=G$ to within about one percent. The theoretical reason for this lack of renormalization[3] consists of assuming that V_α is proportional to the (+) component of the total isotopic spin current $\mathfrak{J}_\alpha$. Then $\partial_\alpha V_\alpha=0$ and remembering that the (isovector) electric current is proportional to the (z) component of $\mathfrak{J}_\alpha$ we can write

$$G\langle p|V_\alpha|n\rangle=G_V\bar{\psi}_p\gamma_\alpha\tau_+\psi_nF_1^V(k^2) -\mu^V(G_V/2M)k_\beta\bar{\psi}_p\sigma_{\alpha\beta}\tau_+\psi_nF_2^V(k^2), \quad (4)$$

where F_1^V, F_2^V are the isovector form factors of the nucleon discussed in Appendix I, and $\mu^V=3.69$.

$(-G_A)/G_V=1.25$, so similar arguments cannot apply in the axial vector case. A theory which is interesting here is that discussed elsewhere.[4,5] For low frequencies, at least, we write

$$\partial_\alpha P_\alpha=(ia/\sqrt{2})\pi^-, \quad (5)$$

where a is a real number. By taking the matrix element of Eq. (5) between neutron and proton states at low momentum transfer we obtain

$$a=-(2M/g_1)m_\pi^2(-G_A/G), \quad (6)$$

where g_1 is the pion-nucleon coupling constant. With this value of a the decay rate of the charged pion can be calculated, resulting in the formula first obtained by Goldberger and Treiman[6] that agrees very well with experiment.

New form factors can be introduced as in reference 5

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[1] B. Pontecorvo, J. Exptl. Theoret. Phys. **37**, 1751 (1959); T. D. Lee and C. N. Yang, Phys. Rev. Letters **4**, 307 (1960); Y. Yamaguchi, Progr. Theoret Phys. (Kyoto) **23**, 1117 (1960); N. Cabibbo and R. Gatto, Nuovo cimento **15**, 159 (1960); S. M. Berman, *International Conference on Theoretical Aspects of Very High-Energy Phenomena, CERN, 1961* (CERN, Geneva, 1961); N. Dombey, High Energy Physics Study report, Lawrence Radiation Laboratory, Berkeley, 1961 (unpublished).

[2] S. Fubini, Y. Nambu, and V. Wataghin, Phys. Rev. **111**, 329 (1958), Paper A; R. Blankenbecler, S. Gartenhaus, R. Huff, and Y. Nambu, Nuovo cimento **17**, 775 (1960), Paper B.

[3] R. P. Feynman and M. Gell-Mann, Phys. Rev. **109**, 193 (1958).

[4] M. Gell-Mann and M. Levy, Nuovo cimento **16**, 705 (1960).

[5] J. Bernstein, S. Fubini, M. Gell-Mann, and W. Thirring, Nuovo cimento **17**, 757 (1960).

[6] M. L. Goldberger and S. B. Treiman, Phys. Rev. **110**, 1178 (1958).

for the axial vector current

$$G\langle p|P_\alpha|n\rangle=(-G_A)\bar{\psi}_p\gamma_\alpha\gamma_5\tau_+\psi_n\alpha(k^2)+ik_\alpha\bar{\psi}_p\gamma_5\tau_+\psi_n\beta(k^2). \quad (7)$$

$\beta(k^2)$ is the induced pseudoscalar term[7] corresponding to Fig. 1 (see Appendix I).

III. INTERMEDIATE BOSON HYPOTHESIS

It is possible that the weak interaction is mediated by a charged spin-one boson W.[8] In such a case one would write

$$\mathfrak{L}_{\mathrm{int}}=g\sqrt{2}J_\alpha\phi_\alpha{}^\dagger+\mathrm{H.c.}, \quad (8)$$

where J_α is the total weak interaction current, ϕ_α is the field operator corresponding to W and g is a coupling constant.

Then, comparing the expression for the amplitude for neutron β decay obtained using Eq. (8) with that obtained from Eq. (2), we obtain

$$2g^2/4\pi=GM_W{}^2/4\pi\sqrt{2}. \quad (9)$$

The effects of the existence of such a particle are exhaustively studied by Lee and Yang.[8] We will tacitly assume that W exists; by using Eq. (9) and then letting $M_W\to\infty$ the results in the current-current theory are

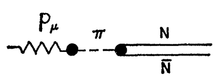


FIG. 1. Induced pseudoscalar term $\beta(k^2)$.

obtained. $W^\pm$ transforms in isotopic spin space just like $\pi^\pm$.

IV. THE SCATTERING AMPLITUDE

We now consider the interaction

$$\nu+N\to l+N+\pi,$$

where l denotes either an electron or muon and we do not distinguish between ν and ν'.

The scattering amplitude for this process is given by

$$T=[-ig^2\sqrt{2}/(k^2+M_W{}^2)]\langle p_2q|j_\mu|p_1\rangle(\delta_{\mu\nu}+k_\mu k_\nu/M_W{}^2)\times\bar{u}(t_2)\gamma_\nu(1+\gamma_5)u(t_1), \quad (10)$$

where p_1 and t_1 are the four-momenta of the initial nucleon and neutrino; p_2, t_2, and q those of the final nucleon, lepton, and pion; j_μ is the total weak current, and $k=t_2-t_1$ is the four-momentum transferred to the pion-nucleon system (or the momentum of the virtual W).

Let us write

$$\mathcal{E}_\mu=i\bar{u}(t_2)\gamma_\mu(1+\gamma_5)u(t_1) \quad (11)$$

and

$$H_\mu=\langle p_2q|j_\mu|p_1\rangle \quad (12)$$

[7] M. L. Goldberger and S. B. Treiman, Phys. Rev. **111**, 354 (1958).

[8] T. D. Lee and C. N. Yang, Phys. Rev. **119**, 1410 (1960).

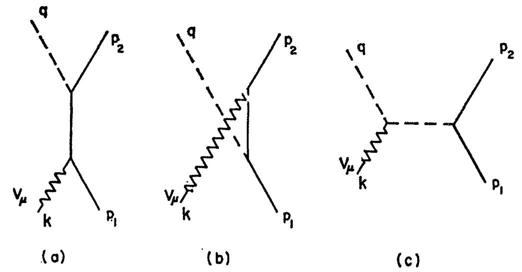


FIG. 2. One-particle intermediate-state diagrams, vector part.

is the matrix element we want to evaluate. For convenience we will write

$$\mathfrak{M}=H\cdot e=\mathfrak{M}_V+\mathfrak{M}_A, \quad (13)$$

where e_μ is an arbitrary four-vector, and $\mathfrak{M}_V$, $\mathfrak{M}_A$ are the parts of the matrix element coming from the vector and axial vector currents.

In the case of the final lepton being an electron, and neglecting the mass of the electron as an "electrical mass difference" between the mass of it and that of the neutrino, similar to the mass difference between the proton and neutron (which we also neglect), we have

$$k\cdot\mathcal{E}=0. \quad (14)$$

Thus, in this case

$$T=-g^2\sqrt{2}H\cdot\mathcal{E}/(k^2+M_W{}^2) \quad (15)$$

and

$$\mathcal{E}_0=\mathbf{k}\cdot\boldsymbol{\varepsilon}/k_0, \quad (16)$$

a relation which is useful in calculations. The amplitude T is a function of the scalars

$$\nu=-P\cdot k/M,\quad \nu_B=q\cdot k/2M,\quad \lambda^2=k^2,\quad P=\tfrac{1}{2}(p_1+p_2). \quad (17)$$

The isotopic dependence of M is just like it is in pion-nucleon scattering; that is, let β be the isotopic state of the W and α that of the pion ($\alpha,\beta=1, 2, 3$).

Then

$$O_i{}^{\alpha\beta}=\delta_{\alpha\beta}O_i{}^++\tfrac{1}{2}[\tau_\alpha,\tau_\beta]O_i{}^-. \quad (18)$$

In terms of total isotopic spin

$$O_i{}^+=\tfrac{1}{3}(O_i{}^{(1/2)}+2O_i{}^{(3/2)}),\qquad O_i{}^-=\tfrac{1}{3}(O_i{}^{(1/2)}-O_i{}^{(3/2)}), \quad (19)$$

where we have put

$$\mathfrak{M}=\textstyle\sum_i M_iO_i;$$

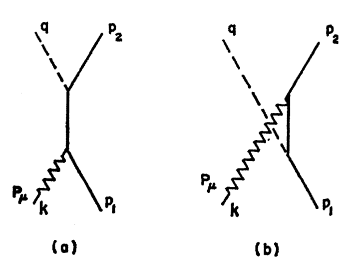


FIG. 3. One-particle intermediate-state diagrams, axial vector part.

M_i are expressions involving gamma-matrices and O_i are invariant scalar amplitudes.

We can first see what general results can be obtained from the theories of the weak currents. First consider

$$\mathfrak{M}_V=\langle p_2q|V_\mu|p_1\rangle e_\mu=H^V\cdot e,$$

together with $\partial_\mu V_\mu=0$.

$$\langle p_2q|\partial_\mu V_\mu|p_1\rangle=0=-ik_\mu\langle p_2q|V_\mu|p_1\rangle,$$

or

$$k\cdot H^V=0. \tag{20}$$

This is analogous to gauge invariance in electromagnetism. Stated formally, gauge invariance asserts here that whenever the vector e_μ is replaced by the momentum k_μ in the amplitude, the amplitude vanishes. We have

$$\mathfrak{M}_V=H^V\cdot e,\quad H^V\cdot k=0.$$

Also, even if $k\cdot\mathcal{E}\neq 0$, the vector part of T is given by

$$T_V=-g^2\sqrt{2}H^V\cdot\mathcal{E}/(k^2+M_W^2). \tag{21}$$

Now consider

$$\mathfrak{M}_A=\langle p_2q|P_\mu|p_1\rangle e_\mu=H^A\cdot e, \tag{22}$$

together with $\partial_\mu\mathbf{P}_\mu\approx ia\boldsymbol{\pi}$. From Eq. (6),

$$a=(-2M/g_1)m_\pi^2(-G_A/G).$$

So

$$\begin{aligned}\langle p_2q|\partial_\mu P_\mu|p_1\rangle&=-ik_\mu\langle p_2q|P_\mu|p_1\rangle\\&=ia\langle p_2q|\pi|p_1\rangle=a\langle p_2q|p_1k\rangle/(k^2+m_\pi^2),\end{aligned} \tag{23}$$

or

$$k\cdot H^A=ia\langle p_2q|p_1k\rangle/(k^2+m_\pi^2),$$

where $\langle p_2q|p_1k\rangle$ is the scattering amplitude for pion-nucleon scattering; the incoming pion having momentum k where k^2 is not necessarily $(-m_\pi^2)$. This relation connects the matrix element for weak pion production with that for pion-nucleon scattering off the mass shell (see Appendix I).

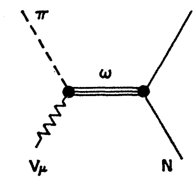
V. DISPERSION RELATIONS

As in A and B we use one-dimensional dispersion relations to calculate the relevant amplitudes, leaving for another time an examination of the problem from the point of view of double dispersion relations. Information about the form of the relations can be obtained from diagrams with one-particle intermediate states (Figs. 2 and 3). Strong two-pion and three-pion interactions can also be similarly considered (Figs. 4 and 5).

The expression for the Born approximation corresponding to Fig. 2 (vector) is

$$\begin{aligned}&g_1[\gamma_5 i\gamma\cdot q/(2p_2\cdot q-1)]\tau_\alpha\tau_\beta[\gamma_\mu F_1{}^V(\lambda^2)\\&\quad-(\mu^V/2M)\sigma_{\mu\nu}k_\nu F_2{}^V(\lambda^2)]e_\mu\\&\quad+g_1[\gamma_\mu F_1{}^V(\lambda^2)-(\mu^V/2M)\sigma_{\mu\nu}k_\nu F_2{}^V(\lambda^2)]\\&\quad\times\tau_\beta\tau_\alpha e_\mu i\gamma\cdot q\gamma_5/(2p_1\cdot q+1)\\&\quad-g_1[\tau_\alpha,\tau_\beta]i\gamma_5 F_\pi(\lambda^2)(2q-k)\cdot e/(2q-k)\cdot k.\end{aligned} \tag{24}$$

FIG. 4. ω exchange.



$F_\pi(\lambda^2)$ is the electromagnetic form factor of the pion (Appendix I) and m_π is taken as unity. Equation (24) is not gauge invariant as it stands. It would be very convenient if it could be made formally gauge invariant as the following calculations essentially use the Born terms as a first approximation; if gauge invariance is not present initially, it would be difficult to impose it on the complete amplitude.

In the case of an electron being produced, $k\cdot\mathcal{E}=0$ and T^V only involves $H^V\cdot\mathcal{E}$. So replacing e_μ by $\mathcal{E}_\mu$ in Eq. (24) and adding

$$g_1[\tau_\alpha,\tau_\beta]i\gamma_5\frac{F_\pi(\lambda^2)-F_1{}^V(\lambda^2)}{\lambda^2}k\cdot\mathcal{E} \tag{25}$$

to it, we have not changed the value of (24) but now it is gauge invariant (cf. A).

If the lepton produced is a muon we cannot do this. Henceforth for simplicity in considering the vector amplitude, we will only treat the electron case.

The Born approximation for M_A (Fig. 3) is

$$\begin{aligned}&g_1[\gamma_5 i\gamma\cdot q/(2p_2\cdot q-1)]\\&\quad\times\tau_\alpha\tau_\beta[\gamma_\mu\gamma_5(-G_A/G)\alpha(\lambda^2)+ik_\mu\gamma_5\beta(\lambda^2)]e_\mu\\&\quad+g_1[\gamma_\mu\gamma_5(-G_A/G)\alpha(\lambda^2)+ik_\mu\gamma_5\beta(\lambda^2)]\\&\qquad\times\tau_\beta\tau_\alpha e_\mu i\gamma\cdot q\gamma_5/(2p_1\cdot q+1).\end{aligned} \tag{26}$$

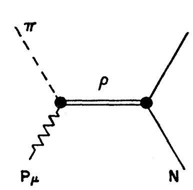
Also we have two-pion (ρ) exchange (Fig. 5) and three-pion (ω) exchange (Fig. 4). We leave out terms involving $V\rho\pi$, $P\omega\pi$ vertices by assuming the GP invariance of the weak current.

For Fig. 4 there is an amplitude

$$\begin{aligned}&\frac{f_{V\omega\pi}F_{V\omega\pi}(\lambda^2)}{(q-k)^2+m_\omega{}^2}\epsilon_{\sigma\rho\mu\nu}k_\sigma q_\rho e_\mu\gamma_{\omega NN}\{\gamma_\nu F_1{}^\omega[(q-k)^2]\\&\qquad+(\mu^S/2M)\sigma_{\nu\lambda}(q-k)_\lambda F_2{}^\omega[(q-k)^2]\}\delta_{\alpha\beta},\end{aligned} \tag{27}$$

where $f_{V\omega\pi}$ and $\gamma_{\omega NN}$ are the renormalized coupling constants for the $V\omega\pi$ and ωNN vertices, $F_{V\omega\pi}(\lambda^2)$ is the form factor for the $V\omega\pi$ vertex, and $F_{1,2}{}^\omega$ are the "charge" and "magnetic moment" form factors for the ωNN vertex. μ^S is just the scalar anomalous magnetic moment of the nucleon (see, however, Appendix I).

FIG. 5. ρ exchange.



The expression (27) can alternatively be written

$$\frac{f_{V\omega\pi}\gamma_{\omega NN}F_{V\omega\pi}(\lambda^2)}{(q-k)^2+m_\omega^2}\bigg\{2\gamma_5[\{P,\gamma\}+\tfrac{1}{2}iM\{\gamma,\gamma\}]F_1^{\omega}[(q-k)^2] - (\mu^S/2M)i\gamma_5\bigg[\tfrac{1}{2}(\lambda^2-4M\nu_B-1)\{\gamma,\gamma\}+\frac{2M\nu_B-\lambda^2}{M\nu_B} \times\{P,q\}-(\nu/\nu_B)\{k,q\}\bigg]F_2^{\omega}[(q-k)^2]\bigg\}\delta_{\alpha\beta}, \quad (28)$$

where

$$\{a,b\}=a\cdot eb\cdot k-a\cdot kb\cdot e.$$

Approximately, we can put $F_{V\omega\pi}(\lambda^2)=1$. Also,

$$f_{V\omega\pi}/g=2f_{\gamma\omega\pi}/e,$$

and $f_{\gamma\omega\pi}$ can be determined from the rate of π^0 decay.[9]

In order to compute the matrix element for Fig. 5 we need to know the amplitude $\langle\pi,\rho|P_\mu|0\rangle$. This would be a useful quantity if it were known, as it is connected to the amplitude for $\pi-\rho$ scattering, and also to the axial vector form factors $\alpha(\lambda^2)$, $\beta(\lambda^2)$. All we know is its pion pole term at present; this gives

$$\frac{-iak\cdot e}{\lambda^2+1}\frac{\gamma_{\pi\pi\rho}F_{\pi\pi\rho}[(q-k)^2]}{(q-k)^2+m_\rho^2}\gamma_{\rho NN}(q+k)_\mu\{\gamma_\mu F_1^{\rho}[(q-k)^2] +(\mu^V/2M)\sigma_{\mu\lambda}(q-k)_\lambda F_2^{\rho}[(q-k)^2]\}[\tau_\alpha,\tau_\beta]. \quad (29)$$

All the quantities appearing in (29) are reasonably well known[9] (Appendix I).

Vector Part

We can write

$$\mathfrak{M}=\sum M_iO_i,$$

where M_i are relativistic invariant forms involving gamma matrices and scalars formed from k, q, e, P; each M_i is linear in e (as we take weak interaction only to first order). There are eight independent M_i allowed, allowing for the Dirac equation for the initial and final nucleon spinors and energy-momentum conservation. O_i are functions of ν, ν_B, λ^2 only and are taken to obey dispersion relations. In the case of $\mathfrak{M}_V$ we have the further requirement of gauge invariance which reduces the number of M_i to six.

As in A take as fundamental forms for $\mathfrak{M}_V$

$$\begin{aligned}
&M_A=\tfrac{1}{2}i\gamma_5\{\gamma,\gamma\}, \quad (+)\\
&M_B=2i\gamma_5\{P,q\}, \quad (+)\\
&M_C=\gamma_5\{\gamma,q\}, \quad (-)\\
&M_D=2\gamma_5[\{\gamma,P\}-\tfrac{1}{2}iM\{\gamma,\gamma\}], \quad (+)\\
&M_E=i\gamma_5\{k,q\}, \quad (-)\\
&M_F=\gamma_5\{k,\gamma\}, \quad (-)
\end{aligned} \quad (30)$$

where $\{a,b\}=a\cdot eb\cdot k-a\cdot kb\cdot e$ is automatically gauge invariant and

$$\mathfrak{M}_V=AM_A+BM_B+\cdots+FM_F. \quad (31)$$

The signs in parentheses in (30) refer to the crossing symmetry of the invariants.

From the isotopic spin decomposition (19) we see that (+) amplitudes are even and (−) are odd under crossing.

The one-dimensional dispersion relations for the energy variable ν, keeping the momentum transfer variable ν_B constant, are

$$\begin{aligned}
A_i^{\pm}(\nu,\nu_B,\lambda^2)&=C_i^{\pm}(\nu,\nu_B,\lambda^2)+R_i^{\pm}(\lambda^2)\left(\frac{1}{\nu_B-\nu}\pm\frac{1}{\nu_B+\nu}\right)\\
&+\frac{1}{\pi}\int_{\nu_0}^{\infty}d\nu'\,\mathrm{Im}A_i^{\pm}(\nu',\nu_B,\lambda^2)\left(\frac{1}{\nu'-\nu}\pm\frac{1}{\nu'+\nu}\right),\\
&\nu_0=\nu_B+1+\frac{1}{2M},\quad i=1,\cdots,6,
\end{aligned} \quad (32)$$

and the $\pm$ sign depends on the crossing symmetry. We are guided to the values of C_i and R_i by the Born approximation (24) together with the additional term (25).

Then we have [compare Eq. (8) of A]

$$\begin{aligned}
&R[A^{\pm}]=-fF_1^V(\lambda^2),\\
&R[B^{\pm}]=fF_1^V(\lambda^2)/2M\nu_B,\\
&R[C^{\pm}]=R[D^{\pm}]=[fF_2^V(\lambda^2)/2M]\mu^V,\\
&R[E^{\pm}]=R[F^{\pm}]=0,
\end{aligned} \quad (33)$$

and

$$\begin{aligned}
&C_A=C_B=C_C=C_D=C_F=0,\\
&C_E^{+}=0,\\
&C_E^{-}=-\frac{4Mf}{\lambda^2}\left(\frac{-2F_\pi(\lambda^2)}{2q\cdot k-\lambda^2}+\frac{F_1^V(\lambda^2)}{q\cdot k}\right).
\end{aligned} \quad (34)$$

C_A^{+}, C_B^{+}, C_D^{+}, and C_E^{+} have contributions from the three-pion intermediate state; these can be read off from Eq. (28).

It has been shown recently that corrections to the original work on the low-energy pion-nucleon problem treated by one-dimensional dispersion relations consist mainly of including pion-pion effects. These effects are useful to an understanding of nonresonant phase shifts; they are not important for the 3–3 resonance. The nonresonant phase shifts were very difficult to observe in pion-nucleon scattering—here we are dealing with a process with a vastly smaller cross section—so it seems reasonable to neglect these ρ- and ω-exchange terms in what follows.

We use the method of Blankenbecler and Gartenhaus[10] to solve the dispersion relations where the 3–3 resonance

[9] M. Gell-Mann, California Institute of Technology Synchrotron Report CTSL-20, 1961 (unpublished). See also: J. J. Sakurai, Ann. Phys. (New York) **11**, 1 (1960); M. Gell-Mann, Phys. Rev. **125**, 1067 (1962).

[10] R. Blankenbecler and S. Gartenhaus, Phys. Rev. **116**, 1297 (1959).

is dominant. This is the approach used in B in the treatment of pion electroproduction and it applies with minor changes for the vector amplitude here. This method treats crossing and recoil exactly, and does not expand in either partial waves or in powers of $1/M$. It is worse to make expansions in $1/M$ in weak production (and electroproduction) than in photoproduction because no longer are there only two terms of order $1/M$ to consider, namely $1/M$ and ω/M ($\omega=W-M$, W = total c.m. energy) which are small. Terms like $\lambda^2/M\omega$ also appear in our case and for wide-angle scattering, which is of importance for the measurement of form factors, this term is not small. It is possible for $\lambda\approx 1$ BeV and still produce a resonant pion-nucleon final state.

The method assumes the phases of the amplitudes in the dispersion relation known. Then the dispersion relations can be solved formally, and a first approximation which could be iterated if necessary to obtain a better approximation is given by

$$A_i(x,\nu_B,\lambda^2) = A_i^{\mathrm{B.A.}}(x,\nu_B,\lambda^2)+\frac{e^{i\delta_{33}(x)}}{\pi}\int_{1+1/2M}^{\infty} dy\,\sin\delta_{33}(y) \times a_i(y,\nu_B,\lambda^2)e^{\Delta(y,x,\nu_B)}\left(\frac{1}{y-x-i\epsilon}\pm\frac{1}{y+x+2\nu_B}\right), \quad (35)$$

where

$$x=\nu-\nu_B=(W^2-M^2)/2M,$$

$\delta_{33}(x)$ is the 3–3 phase shift,

$$\Delta(x',x,\nu_B)=\rho(x,\nu_B)-\rho(x',\nu_B),$$

$$\rho(x,\nu_B)=\frac{P}{\pi}\int_{1+1/2M}^{\infty} dy\,\delta_{33}(y)\times\left[\frac{1}{y-x}+\frac{1}{y+x+2\nu_B}\right], \quad (36)$$

$a_i(x,\nu_B,\lambda^2)$ is the 3–3 projection of $A_i^{\mathrm{B.A.}}(x,\nu_B,\lambda^2)$,

and B.A. denotes Born approximation. We see that

$$e^{\Delta(y,x,\nu_B)}=e^{\rho(x,\nu_B)-\rho(y,\nu_B)},$$

and for x in the resonance region, this function can be expanded in the form

$$e^{\Delta}=1+a(y-x)+\cdots,$$

where under the integral the second term in the expansion is small compared with the first, as the integral is sharply peeked around $y=x_r$.

So now our enhancement term is approximately

$$ie^{i\delta_{33}(x)}a_i(x)\sin\delta_{33}(x)+\frac{e^{i\delta_{33}(x)}}{\pi}\times P\int_{1+1/2M}^{\infty} dy\sin\delta_{33}(y)a_i(y)\left(\frac{1}{y-x}\pm\frac{1}{y+x+2\nu_B}\right). \quad (37)$$

Now the first term gives the right phase as demanded by unitarity to the result; when put with the 3–3 part of $A_i^{\mathrm{B.A.}}$, one gets

$$e^{i\delta_{33}(x)}a_i(x)\sin\delta_{33}(x)\cos\delta_{33}(x)$$

which vanishes at resonance. Also the crossed term under the integral is small in this region and so around resonance we have

$$A_i(x,\nu_B,\lambda^2)=A_i^{\mathrm{B.A.}}(x,\nu_B,\lambda^2)+\frac{e^{i\delta_{33}(x)}}{\pi}P\int_{1+1/2M}^{\infty} dy\frac{a_i(y)\sin\delta_{33}(y)}{y-x}, \quad (38)$$

The principal value integral evidently gives the enhancement to the 3–3 state, and in general will give the enhancement for a resonance with width Γ, say. But we know that a Breit-Wigner expression satisfies the dispersion equations in the resonance region approximately; hence we must expect that Eq. (38) simulates

$$A_i(x,\nu_B,\lambda^2)=A_i^{\mathrm{B.A.}}(x,\nu_B,\lambda^2)+\frac{a_i(x,\nu_B,\lambda^2)}{1-(x/x_r)-i\Gamma}, \quad (39)$$

or even in the resonance region

$$A_i(x,\nu_B,\lambda^2)=\frac{a_1(x,\nu_B,\lambda^2)}{1-(x/x_r)-i\Gamma}, \quad (40)$$

that is, just the Born approximation to go into the 3–3 state with an enhancement factor.

So all we have to do to use any of these expressions for the complete amplitude is to calculate the functions $a_i(x,\nu_B,\lambda^2)$.

We must project out the $\frac{3}{2}$ spin states of $A_i^{\pm}(x,\nu_B,\lambda^2)$. This is done in B. We write the matrix element $\mathfrak{M}_V=\sum AM_A$ in terms of two-component spinors

$$\bar{u}_f\mathfrak{M}_V u_i=\chi_f^*\mathfrak{F}_V\chi_i$$

where

$$\mathfrak{F}_V=\sum_{i=1}^{6}\mathfrak{F}_i\Sigma_i \quad (41)$$

and Σ_i are defined by B.

$$\begin{aligned}\Sigma_1&=i\boldsymbol{\sigma}\cdot\mathbf{a}, & \Sigma_2&=\boldsymbol{\sigma}\cdot\mathbf{q}\boldsymbol{\sigma}\cdot(\mathbf{k}\times\mathbf{a})/qk,\\ \Sigma_3&=i\boldsymbol{\sigma}\cdot\mathbf{k}\mathbf{q}\cdot\mathbf{a}/qk, & \Sigma_4&=i\boldsymbol{\sigma}\cdot\mathbf{q}\mathbf{q}\cdot\mathbf{a}/q^2,\\ \Sigma_5&=i\boldsymbol{\sigma}\cdot\mathbf{k}\mathbf{k}\cdot\mathbf{a}/k^2, & \Sigma_6&=i\boldsymbol{\sigma}\cdot\mathbf{q}\mathbf{k}\cdot\mathbf{a}/qk,\end{aligned} \quad (42)$$

where $\mathbf{a}$ is the gauge-invariant three-vector given by

$$\mathbf{a}=\mathbf{e}-(e_0/k_0)\mathbf{k} \quad (43)$$

and $q=|\mathbf{q}|$, $k=|\mathbf{k}|$, etc. The $\mathfrak{F}_i$ and $A, B, \cdots F$ are related by a set of six linear equations [Eq. (9) of B].

Now writing the isotopic $\frac{3}{2}$ part of $A_i^{\mathrm{B.A.}}(x,\nu_B,\lambda^2)$ in

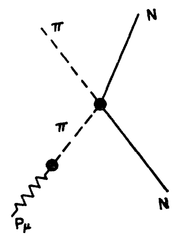


FIG. 6. Main contribution to axial vector amplitude from terms linear in $\beta(\lambda^2)$.

terms of $F_1\cdots F_6$ [convenient multiples of $\mathfrak{F}_1, \cdots\mathfrak{F}_6$; see B Eq. (9)].

$$
\begin{aligned}
F_\mu{}^{\mathrm{B.A.}} &= -\frac{2f}{qk}\mu_V{}^c(\lambda^2)\frac{1}{\cos\theta+a} \\
&\quad\times\left(M,\, -M,\, 1,\, 1,\, \frac{\omega}{2},\, \frac{(W+M)(E_1-M)}{2(E_1+M)}\right), \\
F_g{}^{\mathrm{B.A.}} &= \frac{2f}{qk}F_1{}^V(\lambda^2)\frac{1}{\cos\theta+a} \\
&\quad\times\left(0,\, 0,\, \frac{\omega}{W+M},\, \frac{W+M}{\omega},\, \frac{\omega}{2},\, \frac{W+M}{2}\right), \\
F_\pi{}^{\mathrm{B.A.}} &= \frac{2f}{qk}F_\pi(\lambda^2)\frac{1}{\cos\theta-b} \\
&\quad\times\left(0,\, 0,\, \frac{2M}{W+M},\, -\frac{2M}{\omega},\, -M,\, M\right),
\end{aligned}
\tag{44}
$$

where we have split up the results into terms linear in

$$\mu_V{}^c(\lambda^2),\ F_1{}^V(\lambda^2),\ F_\pi(\lambda^2).$$

Further

$$
\begin{aligned}
&\mathbf{q}\cdot\mathbf{k} = qk\cos\theta, \\
&\mu_V{}^c(\lambda^2) = \mu^V F_2{}^V(\lambda^2) + F_1{}^V(\lambda^2), \\
&\qquad a = (2k_0E_2+\lambda^2)/2qk, \quad b = (2q_0k_0+\lambda^2)/2qk,
\end{aligned}
\tag{45}
$$

where E_1, E_2 are the initial and final nucleon energies, q_0 is the pion energy, and k_0 is the energy of the virtual W.

Now the spin $\frac{3}{2}$ states can be projected out of Eq. (44), leading to expressions of $F_\mu{}^i$, $F_g{}^i$, $F_\pi{}^i$ $(i=1\cdots 6)$. These quantities are given by Eqs. (12)–(15) of B provided that we replace μ_V, e^V, e in B by $\mu_V{}^c(\lambda^2)/M$, $2F_1{}^V(\lambda^2)$, $2F_\pi(\lambda^2)$, respectively.

So the 3–3 projections $a_i(x,\nu_B,\lambda^2)$ are found by taking these equations, substituting them on the left-hand side of B Eq. (9) and solving for $A,\cdots F$. Finally $a^+{}_i=\frac{2}{3}a_i$, $a_i{}^-=-\frac{1}{3}a_i$ in the expression (35).

The Axial Vector Part

We repeat the procedure for $\mathfrak{M}_A$. We no longer have gauge invariance, so this time there are eight invariant amplitudes

$$-i\mathfrak{M}_A = AM_A+\cdots+HM_H, \tag{46}$$

where this time

$$
\begin{aligned}
&M_A = \tfrac{1}{2}(\gamma\cdot q\gamma\cdot e-\gamma\cdot e\gamma\cdot q), && (-) \\
&M_B = 2P\cdot e, && (-) \\
&M_C = q\cdot e, && (+) \\
&M_D = iM\gamma\cdot e, && (-) \\
&M_E = i\gamma\cdot k2P\cdot e, && (+) \\
&M_F = i\gamma\cdot kq\cdot e, && (-) \\
&M_G = k\cdot e, && (+) \\
&M_H = i\gamma\cdot kk\cdot e. && (-)
\end{aligned}
\tag{47}
$$

From the Born approximation (26) we find the residues for the dispersion relations for the $A,\cdots H$ as follows:

$$
\begin{aligned}
R[A^\pm] &= f(-G_A/G)\alpha(\lambda^2), \\
R[C^\pm] &= f(-G_A/G)\alpha(\lambda^2), \\
R[H^\pm] &= -f\beta(\lambda^2).
\end{aligned}
\tag{48}
$$

All other R's are zero, and there are no subtraction terms C_i. If the two-pion intermediate state had been included, we see from Eq. (29) that a subtraction in at least H^- would have been necessary.

In the case of $k\cdot\mathcal{E}$ being zero, amplitudes G and H do not contribute. We will project out the 3–3 states as before.

In terms of two-component spinors,

$$\bar{u}_f\mathfrak{M}_Au_i = i\chi_f{}^*\mathfrak{F}_A\chi_i,$$

where

$$\mathfrak{F}_A = \sum_{i=1}^{8}\mathfrak{F}_i\Sigma_i \tag{49}$$

and

$$
\begin{aligned}
&\Sigma_1 = \boldsymbol{\sigma}\cdot\mathbf{q}\boldsymbol{\sigma}\cdot\mathbf{e}/q, && \Sigma_2 = \mathbf{k}\cdot\mathbf{e}\boldsymbol{\sigma}\cdot\mathbf{q}\boldsymbol{\sigma}\cdot\mathbf{k}/qk^2, \\
&\Sigma_3 = \mathbf{q}\cdot\mathbf{e}\boldsymbol{\sigma}\cdot\mathbf{q}\boldsymbol{\sigma}\cdot\mathbf{k}/q^2k, && \Sigma_4 = (\boldsymbol{\sigma}\cdot\mathbf{q}\boldsymbol{\sigma}\cdot\mathbf{k}/qk)e_0, \\
&\Sigma_5 = \boldsymbol{\sigma}\cdot\mathbf{e}\boldsymbol{\sigma}\cdot\mathbf{k}/k, && \Sigma_6 = \mathbf{q}\cdot\mathbf{e}/q, \\
&\Sigma_7 = \mathbf{k}\cdot\mathbf{e}/k, && \Sigma_8 = e_0.
\end{aligned}
\tag{50}
$$

Then the relations for the $\mathfrak{F}$'s in terms of $A,\cdots H$ are

$$
\begin{aligned}
F_1 &= (2M/qO_2)\mathfrak{F}_1 = (W+M)A-MD, \\
F_2 &= (2MO_1/qk^2)\mathfrak{F}_2 = B+(W+M)E-G-(W+M)H, \\
F_3 &= (2MO_1/q^2k)\mathfrak{F}_3 = A+B-C \\
&\qquad\qquad +(W+M)E-(W+M)F, \\
F_4 &= (2MO_1/qk)\mathfrak{F}_4 = (E_2+M)A+(E_1+E_2)B \\
&\qquad\qquad +q_0C-MD+k_0G+(W+M) \\
&\qquad\qquad \times[(E_1+E_2)E+q_0F+k_0H], \\
F_5 &= (2MO_2/k)\mathfrak{F}_5 = -\omega A-MD, \\
F_6 &= (2M/qO_1)\mathfrak{F}_6 = -A-B+C+\omega(E-F), \\
F_7 &= (2M/kO_1)\mathfrak{F}_7 = -B+\omega E+G-\omega H, \\
F_8 &= (2M/O_1)\mathfrak{F}_8 = -(E_2-M)A-(E_1+E_2)B \\
&\qquad\qquad -q_0C-MD-k_0G \\
&\qquad\qquad +\omega[(E_1+E_2)E+q_0F+k_0H], \\
O_1 &= [(E_1+M)(E_2+M)]^{\frac{1}{2}}, \quad O_2 = [(E_1+M)/(E_2+M)]^{\frac{1}{2}}.
\end{aligned}
\tag{51}
$$

When $k\cdot\mathcal{E}=0$, F_8 becomes incorporated in F_7 and F_4 in F_2. We will project out the 3–3 part of only the terms linear in $\alpha(\lambda^2)$; if $k\cdot\mathcal{E}=0$, these are the only terms of interest, and if not, there is a simple way to relate the β terms to pion-nucleon scattering which will be shown later.

So the isotopic $\frac{3}{2}$ part of the amplitudes are given by

$$F_\alpha^{\text{B.A.}}=\frac{-2Mf\alpha(\lambda^2)(-G_A/G)}{qk(a+\cos\theta)}[W+M,0,2,E_2+M-q_0,\\ -\omega,-2,0,-(E_2-q_0-M)]. \quad (52)$$

Doing the spin $\frac{3}{2}$ projections, we obtain

$$\frac{F_\alpha^1}{A_\alpha}=-\omega\frac{k(E_2+M)}{q(E_1+M)}\bar{\alpha}(a)+(E_2+M)\bar{\beta}(a)+\frac{W+M}{2}\bar{\gamma}(a),$$

$$\frac{F_\alpha^2}{A_\alpha}=\frac{2\omega(E_2+M)}{qk}\bar{\alpha}(a)-\left(\frac{q}{k}a+\frac{E_2+M}{E_1-M}\right)\bar{\gamma}(a),$$

$$F_\alpha^3=0,$$

$$F_\alpha^4/A_\alpha=[(E_1+M)(E_2+M)/qk](E_2-q_0-M)\bar{\alpha}(a)+\tfrac{1}{2}(E_2+M-q_0)\bar{\gamma}(a),$$

$$F_\alpha^5/A_\alpha\cos\theta=-3\omega\bar{\alpha}(a)+2(E_2-M)[1-\tfrac{3}{2}a\bar{\beta}(a)]+\frac{3}{2}\frac{q}{k}\frac{E_1+M}{E_2+M}(W+M)\bar{\gamma}(a), \quad (53)$$

$$F_\alpha^6/A_\alpha=-3\bar{\beta}(a)+[2qk/(E_1+M)(E_2+M)]\times[1-\tfrac{3}{2}a\bar{\beta}(a)],$$

$$F_\alpha^7/A_\alpha\cos\theta=[-3qq_0/k(E_2+M)]\bar{\gamma}(a)+6[(E_2-M)/(E_1+M)]f(a),$$

$$F_\alpha^8/A_\alpha\cos\theta=-3(E_2-M-q_0)\bar{\alpha}(a)-\tfrac{3}{2}[qk/(E_1+M)(E_2+M)]\bar{\gamma}(a),$$

$$A_\alpha=-(2Mf/qk)(-G_A/G)\alpha(\lambda^2).$$

Here

$$\begin{aligned}\bar{\alpha}(a)&=1-\tfrac{1}{2}a\ln[(a+1)/(a-1)],\\ \bar{\beta}(a)&=a-\tfrac{1}{2}(a^2-1)\ln[(a+1)/(a-1)],\\ \bar{\gamma}(a)&=3a-\tfrac{1}{2}(3a^2-1)\ln[(a+1)/(a-1)],\\ f(a)&=\tfrac{1}{3}+a^2\bar{\alpha}(a)-[1-\tfrac{3}{2}a\bar{\beta}(a)].\end{aligned} \quad (54)$$

As before, Eqs. (51) must now be solved for $A, B, \cdots H$ in terms of $F_\alpha^1, \cdots F_\alpha^8$. We now have to compute the terms linear in $\beta(\lambda^2)$. These are not present in a situation where $k\cdot\mathcal{E}=0$, as in weak pion production with electron. But if a muon is produced they are, and could be appreciable for high momentum transfer. We could compute them by the same methods that have been used up to now, but there is a simple way to relate any term involving $\beta(\lambda^2)$ to a similar term involving a pion which can be used more generally than the other methods (the dominance of a resonance is not required).

In our case, consider the diagram Fig. 6. For it, we have

$$\langle qp_2|P_\mu|p_1\rangle_\beta=\langle qp_2|\pi_k|p_1\rangle\langle\pi_k|P_\mu|0\rangle.$$

But

$$\langle qp_2|\pi_k|p_1\rangle=[-i/(\lambda^2+m_\pi^2)]\langle qp_2|kp_1\rangle,$$

where, as in Eq. (23), $\langle qp_2|kp_1\rangle$ describes pion-nucleon scattering with the initial pion off the mass shell.

Also, from the theory of the axial vector current, for λ^2 not too large,

$$\begin{aligned}\langle\pi_k|P_\mu|0\rangle&=(-ik_\mu/m_\pi^2)\langle\pi|\partial_\mu P_\mu|0\rangle\\ &=a(k_\mu/m_\pi^2)\langle\pi|\pi(x)|0\rangle.\end{aligned}$$

The last matrix element is just a phase factor which we can take to equal to one. So

$$\begin{aligned}H_\beta&=\langle qp_2|P_\mu|p_1\rangle_\beta e_\mu\\ &=-(ia/m_\pi^2)k\cdot e\langle qp_2|kp_1\rangle/(\lambda^2+m_\pi^2).\end{aligned} \quad (55)$$

But

$$\beta(\lambda^2)=(ag_1/m_\pi^2)/(\lambda^2+m_\pi^2)+\cdots.$$

So

$$H_\beta=-[i\beta(\lambda^2)/g_1]k\cdot e\langle qp_2|kp_1\rangle. \quad (56)$$

Now

$$\langle qp_2|kp_1\rangle=\bar{u}_f(-A+i\gamma\cdot kB)u_i,$$

where

$$A=A(\nu,\nu_B,\lambda^2),\quad B=B(\nu,\nu_B,\lambda^2).$$

For $\lambda^2=-m_\pi^2$, we know that there are no one-nucleon poles in A and that the residue for these poles in B is $g_1^2/2M$. So we expect no poles here for our amplitude G, and a pole of residue

$$-[i\beta(\lambda^2)/g_1]g_1^2/2M=-i\beta(\lambda^2)f \quad \text{for} \quad H$$

[Eq. (48)], thus showing our choice of phase factor is correct. Let us put

$$i\langle p_2q|\pi_k|p_1\rangle=\bar{u}_f(-A_0+i\gamma\cdot kB_0)u_i=\frac{\bar{u}_f(-A+i\gamma\cdot kB)u_i}{\lambda^2+m_\pi^2}. \quad (57)$$

We expect A_0, B_0 to be analytic in ν, ν_B, and λ^2. Write new variables,

$$s=-(p_1+k)^2,\quad t=-(p_1-p_2)^2,$$

instead of ν, ν_B. Then for $s<(M+1)^2$, $t<9$; $A_0(s,t,\lambda^2)$ and $B_0(s,t,\lambda^2)$ are real for $(-\lambda^2)<9$ and we expect the following dispersion equations to hold:

$$A_0(s,t,\lambda^2)=\frac{A(s,\,t,\,-m_\pi^2)}{\lambda^2+m_\pi^2}+\frac{1}{\pi}\int_9^\infty\frac{\mathrm{Im}A_0(s,t,\lambda'^2)d\lambda'^2}{\lambda'^2+\lambda^2},$$

$$B_0(s,t,\lambda^2)=\frac{B(s,\,t,\,-m_\pi^2)}{\lambda^2+m_\pi^2}+\frac{1}{\pi}\int_9^\infty\frac{\mathrm{Im}B_0(s,t,\lambda'^2)d\lambda'^2}{\lambda'^2+\lambda^2}. \tag{58}$$

In the same way that we discuss the functions $\beta(\lambda^2)$, $K(\lambda^2)$ (Appendix I) we expect for reasonably small λ^2 to be able to neglect the integrals in these equations in comparison with the pion pole terms.

Now let us continue analytically in s to the resonance region; $s\approx(M+2)^2$. The functions A, B immediately become complex as s becomes greater than $(M+1)^2$, and presumably so do the continuations of $\mathrm{Im}A_0$, $\mathrm{Im}B_0$. We know that A and B contain a large imaginary part in the resonance region, but there is no reason to expect either the real or imaginary parts of $\mathrm{Im}A_0$, $\mathrm{Im}B_0$ to become appreciably larger for these values of s than they were before. We have not continued very far. So the obvious approximation is to write

$$A_0=A(s,\,t,\,-m_\pi^2)/(\lambda^2+m_\pi^2),$$
$$B_0=B(s,\,t,\,-m_\pi^2)/(\lambda^2+m_\pi^2). \tag{59}$$

So

$$H_\beta=-[i\beta(\lambda^2)/g_1]k\cdot e\bar{u}_f[-A(\nu,\,\nu_B,\,-m_\pi^2)+i\gamma\cdot kB(\nu,\,\nu_B,\,-m_\pi^2)]. \tag{60}$$

Now from the study of the pion-nucleon problem by CGLN,[11] we have

$$A^{33}=\left(3\frac{W+M}{E_2+M}\cos\theta+\frac{\omega}{E_2-M}\right)\frac{4\pi e^{i\delta_{33}}\sin\delta_{33}}{q},$$

$$B^{33}=\left(\frac{3}{E_2+M}\cos\theta-\frac{1}{E_2-M}\right)\frac{4\pi e^{i\delta_{33}}\sin\delta_{33}}{q}. \tag{61}$$

VI. STATIC LIMIT

The simplest way to see what Eq. (9) of B and Eq. (53) are about is to go to the static limit, expand in ω/M, and keep λ^2 small.

We have three sets of terms $\mathfrak{F}_\mu$, $\mathfrak{F}_g$, $\mathfrak{F}_\pi$ for the vector amplitude. The charge terms $\mathfrak{F}_g$ are well known to be essentially recoil terms and can be neglected in the static limit. For the terms $\mathfrak{F}_\mu$, $\mathfrak{F}_\mu^4=0$, $\mathfrak{F}_\mu^5$ and $\mathfrak{F}_\mu^6$ are small if λ^2 is small. So we are left with $\mathfrak{F}_\mu^1$, $\mathfrak{F}_\mu^2$, and $\mathfrak{F}_\mu^3$.

Explicitly the leading terms in powers of ω/M (and $1/M$) are given by

$$\mathfrak{F}_\mu^1/A_\mu\cos\theta=3M\omega\bar{\alpha}(a),$$
$$\mathfrak{F}_\mu^2/A_\mu=-\tfrac{1}{2}qk\bar{\beta}(a)+M\omega\bar{\alpha}(a), \tag{62}$$
$$\mathfrak{F}_\mu^3/A_\mu=\tfrac{3}{2}qk\bar{\beta}(a).$$

To first order,

$$\bar{\alpha}(a)=-\tfrac{3}{4}[\bar{\beta}(a)]^2.$$

So to zeroth order,

$$\mathfrak{F}_\mu^1/A_\mu\cos\theta=-\tfrac{3}{2}qk\bar{\beta}(a),$$
$$\mathfrak{F}_\mu^2/A_\mu=-qk\bar{\beta}(a).$$

So

$$\mathfrak{F}_\mu^{33}/A_\mu=[-\tfrac{3}{2}i\boldsymbol{\sigma}\cdot\mathbf{a}\mathbf{q}\cdot\mathbf{k}-\boldsymbol{\sigma}\cdot\mathbf{q}\boldsymbol{\sigma}\cdot(\mathbf{k}\times\mathbf{a})+\tfrac{3}{2}i\boldsymbol{\sigma}\cdot\mathbf{k}\mathbf{q}\cdot\mathbf{a}]\bar{\beta}(a).$$

If we talk about a weak pion production with electrons, then

$$\mathbf{a}=\boldsymbol{\varepsilon}-(\mathbf{k}\cdot\boldsymbol{\varepsilon}/k_0^2)\mathbf{k},$$

so

$$\mathfrak{F}_\mu^{33}/A_\mu=-[2(\mathbf{q}\times\mathbf{k})\cdot\boldsymbol{\varepsilon}+i\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}\mathbf{q}\cdot\mathbf{k}-i\boldsymbol{\sigma}\cdot\mathbf{k}\mathbf{q}\cdot\boldsymbol{\varepsilon}]\bar{\beta}(a)/2. \tag{63}$$

Notice that the longitudinal terms have dropped out. It is clear that this term is the magnetic dipole term. Indeed $\bar{\beta}(a)$ is a multiple of the function termed F_M by CGLN.

Finally, then,

$$\begin{aligned}\mathfrak{F}_\mu^{33}&=[f\mu_V{}^c(\lambda^2)/qk]\bar{\beta}(a)\\&\quad\times[2(\mathbf{q}\times\mathbf{k})\cdot\boldsymbol{\varepsilon}+i\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}\mathbf{q}\cdot\mathbf{k}-i\boldsymbol{\sigma}\cdot\mathbf{k}\mathbf{q}\cdot\boldsymbol{\varepsilon}]\\&=[2f\mu_V{}^c(\lambda^2)/3M\omega]\\&\quad\times[2(\mathbf{q}\times\mathbf{k})\cdot\boldsymbol{\varepsilon}+i\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}\mathbf{q}\cdot\mathbf{k}-i\boldsymbol{\sigma}\cdot\mathbf{k}\mathbf{q}\cdot\boldsymbol{\varepsilon}].\end{aligned} \tag{64}$$

Next there is $\mathfrak{F}_\pi$. These terms are difficult to deal with convincingly. The treatment of photoproduction is fairly satisfactory in its agreement with experiment without including the pionic current terms at resonance. Yet the F_π at first glance appear as big as F_μ. Indeed $F_\mu{}^1$, for example, is of order $1/M$, whereas $F_\pi{}^1$ is of order one. However $1-\frac{3}{2}b\bar{\beta}(b)$ is fairly small although it does not involve M, and it turns out that $F_\mu{}^1$ is greater than $F_\pi{}^1$ by about a factor of $\frac{3}{2}$ for the region we are interested in. Also, because of the factor $1-\frac{3}{2}b\bar{\beta}(b)$, $F_\pi{}^1$ and even more so $\mathfrak{F}_\pi{}^1$ is a decreasing function of ω, whereas $\mathfrak{F}_\mu{}^1$ is roughly constant in ω. The same considerations apply to $\mathfrak{F}^2$ and $\mathfrak{F}^3$. In view of this the enhancement integral,

$$P\int_{1+1/2M}^\infty\frac{a_i(y)\sin\delta_{33}(y)}{y-x}dy,\quad y=\frac{\omega^2+2M\omega}{2M}, \tag{38}$$

should contribute rather less to the pionic terms than to the magnetic terms. Finally, and most important,

$$A_\mu=-(2f/qk)\mu_V{}^c(\lambda^2),\quad B_\pi=(2f/qk)F_\pi(\lambda^2)$$

and so A_μ is about 4.7 times as large as B_π. In view of all

[11] G. F. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu, Phys. Rev. **106**, 1337 and 1345 (1957).

this, it should be no worse to neglect the pionic terms than to neglect terms of order $1/M$.

So the simplest approximation to $\mathfrak{F}_V{}^{33}$ is just the magnetic dipole term.

$$\mathfrak{F}_V{}^{33}=\mathfrak{F}_\mu{}^{33}=\frac{2f\mu_V{}^c(\lambda^2)}{3M\omega}\times[2(\mathbf{q}\times\mathbf{k})\cdot\boldsymbol{\mathcal{E}}+i\boldsymbol{\sigma}\cdot\boldsymbol{\mathcal{E}}\mathbf{q}\cdot\mathbf{k}-i\boldsymbol{\sigma}\cdot\mathbf{k}\mathbf{q}\cdot\boldsymbol{\mathcal{E}}]. \quad (65)$$

Next is $\mathfrak{F}_A$. We are considering the case where $k\cdot\mathcal{E}=0$ so the terms in $\beta(\lambda^2)$ vanish. By inspection of Eq. (51) and (53) we see that the largest terms are

$$\mathfrak{F}_\alpha{}^1/A_\alpha=q\bar{\beta}(a)$$

and

$$\mathfrak{F}_\alpha{}^6/A_\alpha=-3q\bar{\beta}(a), \quad (66)$$

to zeroth order in $1/M$. So

$$\mathfrak{F}_\alpha{}^{33}/A_\alpha=-(3\mathbf{q}\cdot\boldsymbol{\mathcal{E}}-\boldsymbol{\sigma}\cdot\mathbf{q}\boldsymbol{\sigma}\cdot\boldsymbol{\mathcal{E}})\bar{\beta}(a) \quad (67)$$

and

$$\mathfrak{F}_A{}^{33}=\mathfrak{F}_\alpha{}^{33}=\frac{4f(-G_A/G)\alpha(\lambda^2)}{3\omega}(3\mathbf{q}\cdot\boldsymbol{\mathcal{E}}-\boldsymbol{\sigma}\cdot\mathbf{q}\boldsymbol{\sigma}\cdot\boldsymbol{\mathcal{E}}). \quad (68)$$

$\mathfrak{F}_A{}^{33}$ looks appreciably larger than $\mathfrak{F}_V{}^{33}$. This is not surprising, as the direct vector weak interaction, the charge term, does not contribute in the static limit to a spin $\frac{3}{2}$ parity (+) state of the final system; it is the weak magnetism which contributes. On the other hand, the axial vector term can go directly in the static limit.

The next task is to find the actual amplitudes to be used in a calculation. If we are interested only in the resonance region we can drop both the Born terms and the crossed term in the complete expressions for the amplitudes, and we can write the resonant term in the form

$$A_i(x,\nu_B,\lambda^2)=\frac{a_i(x,\nu_B,\lambda^2)}{1-(x/x_r)-i\Gamma}, \quad (40)$$

where Γ refers to the width of the resonance. Now in the static limit $x\approx\omega$, and we recall the Chew-Low formula for the enhancement factor for the pion-nucleon system in the static limit.[12]

$$e^{i\delta_{33}}\sin\delta_{33}=\frac{\frac{4}{3}(f^2/4\pi)q^3/\omega}{1-(\omega/\omega_r)-i\frac{4}{3}(f^2/4\pi)q^3/\omega}. \quad (69)$$

So we can write

$$A_i(\omega,\nu_B,\lambda^2)=\frac{a_i(\omega,\nu_B,\lambda^2)}{\frac{4}{3}(f^2/4\pi)q^3/\omega}e^{i\delta_{33}}\sin\delta_{33}. \quad (70)$$

Now the A_i are linear in the F_i and so our final expressions will be

$$\mathfrak{F}_V=\frac{2f\mu_V{}^c(\lambda^2)}{3M\omega}\frac{1}{\frac{4}{3}(f^2/4\pi)q^3/\omega}[2(\mathbf{q}\times\mathbf{k})\cdot\boldsymbol{\mathcal{E}}+i\boldsymbol{\sigma}\cdot\boldsymbol{\mathcal{E}}\mathbf{q}\cdot\mathbf{k}-i\boldsymbol{\sigma}\cdot\mathbf{k}\mathbf{q}\cdot\boldsymbol{\mathcal{E}}]e^{i\delta_{33}}\sin\delta_{33}. \quad (71)$$

[12] G. F. Chew and F. E. Low, Phys. Rev. **101**, 1570 (1956).

This expression is given in A. Also,

$$\mathfrak{F}_A=\frac{4f(-G_A/G)\alpha(\lambda^2)}{3\omega}\frac{1}{\frac{4}{3}(f^2/4\pi)q^3/\omega}\times(3\mathbf{q}\cdot\boldsymbol{\mathcal{E}}-\boldsymbol{\sigma}\cdot\mathbf{q}\boldsymbol{\sigma}\cdot\boldsymbol{\mathcal{E}})e^{i\delta_{33}}\sin\delta_{33}. \quad (72)$$

Let us write

$$M_+=2\pi\mu_V{}^c(\lambda^2)/fMq^3,\quad M_-=4\pi(-G_A/G)\alpha(\lambda^2)/fq^3. \quad (73)$$

VII. CROSS SECTION CALCULATIONS

In general, for weak pion production with electron

$$\mathfrak{F}_V=\boldsymbol{\mathfrak{F}}_V\cdot\boldsymbol{\mathcal{E}},\quad \mathfrak{F}_A=\boldsymbol{\mathfrak{F}}_A\cdot\boldsymbol{\mathcal{E}},$$

where

$$\boldsymbol{\mathfrak{F}}_V=\mathbf{P}+i\boldsymbol{\sigma}Q+i(\boldsymbol{\sigma}\cdot\mathbf{u}_1\mathbf{q}+\boldsymbol{\sigma}\cdot\mathbf{u}_2\mathbf{k}),$$
$$\boldsymbol{\mathfrak{F}}_A=\mathbf{X}+i(\boldsymbol{\sigma}\cdot\mathbf{v}_1\mathbf{q}+\boldsymbol{\sigma}\cdot\mathbf{v}_2\mathbf{k})+\boldsymbol{\sigma}\cdot\mathbf{Y}\boldsymbol{\sigma}. \quad (74)$$

We can find $\mathbf{P}$, Q, etc. from the definitions of $\mathfrak{F}_V$ and $\mathfrak{F}_A$ in Eqs. (42) and (50).

We write (averaging over initial and summing over final nucleon spin)

$$T_{\alpha\beta}=\tfrac{1}{2}\,\mathrm{Sp}[(\mathfrak{F}_{V\alpha}{}^*-i\mathfrak{F}_{A\alpha}{}^*)(\mathfrak{F}_{V\beta}+i\mathfrak{F}_{A\beta})]. \quad (75)$$

It is now easy to find $T_{\alpha\beta}$ by substituting in Eq. (75) the expressions for $\boldsymbol{\mathfrak{F}}_V$ and $\boldsymbol{\mathfrak{F}}_A$ of Eq. (74).

$T_{\alpha\beta}$ must be contracted with the leptonic contribution,

$$e_{\alpha\beta}=\mathrm{Sp}[t_1\gamma_\alpha(1+\gamma_5)t_2\gamma_\beta(1+\gamma_5)]$$
$$=8[2t_{1\alpha}t_{1\beta}-t_{1\alpha}k_\beta-k_\alpha t_{1\beta}+\tfrac{1}{2}\lambda^2\delta_{\alpha\beta}+\epsilon_{\alpha p\beta q}k_pt_{1q}]. \quad (76)$$

t_1 and t_2 are the four-momenta of the neutrino and electron, respectively, $t_1=\gamma\cdot t_1$, $k=t_1-t_2$, $\epsilon_{\alpha p\beta q}$ is the 4-index completely antisymmetric permutation symbol. $\alpha,\beta=1,2,3$; $p,q=1,2,3,4$. Write

$$|\mathfrak{M}|^2=T_{\alpha\beta}e_{\alpha\beta}. \quad (77)$$

Then

$$\sigma v=\frac{2g^4M^2}{8E_1E_2q_0t_1t_2}\frac{|\mathfrak{M}|^2}{(\lambda^2+M_W{}^2)^2}\times(2\pi)^4\delta^4(p_2+q_2+t_2-t_1-p_1)\frac{d^3p_2d^3qd^3t_2}{(2\pi)^3(2\pi)^3(2\pi)^3}. \quad (78)$$

We will be concerned with an experiment in which the initial nucleon is at rest; that is $E_1=M$ and $v=1$. However, $|\mathfrak{M}|^2$ must be evaluated in the center of mass system of the final nucleon and pion. So let us use lower case letters as before for c.m. quantities, and capital letters for the quantities evaluated in the laboratory frame; i.e., T_1, T_2, K, K_0, P_2, Q, Q_0($E_1{}^{\mathrm{lab}}=M$ and $E_2{}^{\mathrm{lab}}$ never appear).

Then the differential cross section for production of a pion in solid angle $d\Omega_q$ and electron in solid angle $d\Omega_{t_2}$ is

$$\sigma=\frac{g^4M}{128\pi^5Q_0T_1}\frac{|\mathfrak{M}|^2}{(\lambda^2+M_W{}^2)^2}\times\frac{Q^2T_2{}^2d\Omega_qd\Omega_{t_2}dQ}{T_2(T_1+M-Q_0)-\mathbf{T}_2\cdot(\mathbf{T}_1-\mathbf{Q})}. \quad (79)$$

It is possible, following Dalitz and Yennie,[13] to obtain the cross section for inelastic lepton scattering in a simple form. (Inelastic lepton scattering implies that only the final lepton is observed.)

First notice that

$$|\mathfrak{M}|^2(d^3p_2/E_2)(d^3q/q_0)\delta^4(p_2+q-k-p_1) \quad (80)$$

is a Lorentz scalar and hence can be evaluated in any frame; in particular in the center-of-mass frame of the final pion and nucleon. So it is just

$$|\mathfrak{M}|^2(d^3p_2/E_2)(d^3q/q_0)\delta(\mathbf{p}_2+\mathbf{q})\delta(E_2+q_0-k_0-E_1). \quad (81)$$

This can be integrated to

$$(q/W)|\mathfrak{M}|^2d\Omega_q=(4\pi q/W)|\mathfrak{M}|^2d\bar{\Omega}_q, \quad (82)$$

where $d\bar{\Omega}_q$ implies that we are going to average over the directions of the final pion.

Write

$$\langle\mathfrak{M}^2\rangle=\int\frac{4\pi q}{W}|\mathfrak{M}|^2d\bar{\Omega}_q. \quad (83)$$

Then

$$\frac{d^2\sigma}{d\Omega dT_2}=\frac{g^4M}{128\pi^5}\frac{T_2}{T_1}\frac{\langle\mathfrak{M}^2\rangle}{(\lambda^2+M_W^2)^2}. \quad (84)$$

If we are not interested in looking for intermediate boson effects, we remember that

$$g^2=GM_W^2/2\sqrt{2}, \quad (9)$$

so

$$g^4/(\lambda^2+M_W^2)^2\approx G^2/8. \quad (85)$$

Now we have to find the form of $\langle\mathfrak{M}^2\rangle$ in the static limit. Here

$$\begin{aligned}
&\mathbf{P}=2(\mathbf{q}\times\mathbf{k})M_+e^{i\delta_{33}}\sin\delta_{33},\\
&Q=\mathbf{q}\cdot\mathbf{k}M_+e^{i\delta_{33}}\sin\delta_{33},\\
&\mathbf{u}_1=-\mathbf{k}M_+e^{i\delta_{33}}\sin\delta_{33},\quad \mathbf{u}_2=0,\\
&\mathbf{X}=3\mathbf{q}M_-e^{i\delta_{33}}\sin\delta_{33},\\
&\mathbf{Y}=-\mathbf{q}M_-e^{i\delta_{33}}\sin\delta_{33},\\
&\mathbf{v}_1=\mathbf{v}_2=0.
\end{aligned} \quad (86)$$

These are pure $I=\frac{3}{2}$ amplitudes (i.e., they apply directly to $\nu+p\rightarrow e^-+p+\pi^+$).

Averaging over angles, we have

$$\begin{aligned}
\langle\mathfrak{M}^2\rangle=(4\pi q/W)[&16q^2k^2M_+^2(2t_1^2\sin^2\psi+\lambda^2)\\
&+8q^2M_-^2(4t_1^2+3\lambda^2-4t_1k\cos\psi)\\
&+64q^2kt_1M_+M_-(k-k_0\cos\psi)]\sin^2\delta_{33}, \quad (87)
\end{aligned}$$

where ψ is the angle between $\boldsymbol{t}_1$ and $\mathbf{k}$.

VIII. CONCLUSIONS

Using the methods of this paper the amplitude for $\nu+N\rightarrow e+\pi+N$ can be calculated as accurately as desired when the c.m. energy of the pion-nucleon state is such that the 3–3 isobar dominates the situation, provided that the relevant form factors are known. If λ^2 is also small, the static limit should be reasonable to use and the resulting formulae are simple.

At higher incident neutrino energies the pion-exchange "peripheral" term [Fig. 2(c) and Eq. (34)] should be included in the amplitude. It becomes more important as the neutrino energy increases and the importance of the 3–3 isobar decreases.

A measurement of the energy spectrum of the final electron at an appropriate fixed angle would show the resonance peak due to the 3–3 isobar. The height of the peak is determined by the form factors [mainly by $\alpha(\lambda^2)$].

The matrix element $\mathfrak{M}$ [Eq. (13)] describes also the process $\pi+N\rightarrow W+N$. Thus the cross section for this reaction can be found using the preceding analysis (Appendix II).

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APPENDIX I. THE FORM FACTORS

We expect all form factors to satisfy dispersion relations (possibly with subtractions). For example, the nuclear electromagnetic form factors:

$$F_i^S(s)=\frac{1}{\pi}\int_9^\infty\frac{g_i^S(s')ds'}{s'-s}, \quad (A1)$$

$$F_i^V(s)=\frac{1}{\pi}\int_4^\infty\frac{g_i^V(s')ds'}{s'-s}. \quad (A2)$$

$i=1$ or 2, $s=-\lambda^2$, and we have put $m_\pi=1$. Also, the electromagnetic form factor of the pion:

$$F_\pi(s)=\frac{1}{\pi}\int_4^\infty\frac{g_\pi(s')}{s'-s}ds'. \quad (A3)$$

Experimentally strong two-pion and three-pion interactions are observed. We hope that these dominate the dispersion integrals (A1, 2, 3).

Let us deal first with $F_\pi(s)$. Here the $I=1$, $J=1$ pion-pion resonance at 750 MeV with a width of[14] about 100 MeV is expected to play the important role.

Using the language of vector meson theory,[15] we call this unstable particle ρ, with a decay rate into two pions

[13] R. H. Dalitz and D. R. Yennie, Phys. Rev. **105**, 1598 (1957).

[14] A. R. Erwin, R. March, W. D. Walker, and E. West, Phys. Rev. Letters **6**, 628 (1961); D. Stonehill, C. Baltay, H. Courant, W. Fickinger, E. C. Fowler, H. Kraybill, J. Sandweiss, J. Sanford, and H. Taft, *ibid.* **6**, 624 (1961); E. Pickup, D. K. Robinson, and E. O. Salant, *ibid.* **7**, 192 (1961); Bologna, Orsay, and Saclay groups, presented by G. Puppi, Report at the International Conference on Elementary Particles, Aix-en-Provence, 1961 (unpublished).

[15] M. Gell-Mann and F. Zachariasen, Phys. Rev. **124**, 953 (1961).

given by

$$\Gamma_\rho = \frac{1}{3}\frac{\gamma_{\rho\pi\pi}{}^2}{4\pi}\frac{(m_\rho{}^2-4)^{\frac{3}{2}}}{m_\rho{}^2}. \qquad \text{(A4)}$$

Near $s=m_\rho{}^2$, the form factor of the $\rho\pi\pi$ vertex is

$$F_{\rho\pi\pi}(s)=(s-m_\rho{}^2)/(s-m_\rho{}^2+im_\rho\Gamma_\rho). \qquad \text{(A5)}$$

In general,[15] the electromagnetic form factors are related to the ρ form factors by

$$F_{\text{el}}{}^V(s)=\frac{-m_\rho{}^2}{s-m_\rho{}^2}\frac{F_\rho(s)}{F_\rho(0)}, \qquad \text{(A6)}$$

where $F_{\text{el}}{}^V(s)$ is any isotopic vector electromagnetic form factor and $F_\rho(s)$ is the corresponding ρ form factor. According to Sakurai[9] the ρ meson is coupled to a conserved current (the isotopic spin current). So at zero momentum transfer it should have a universal interaction with the isotopic spin current. This can be expressed by

$$\gamma_\rho=\gamma_{\rho\pi\pi}F_{\rho\pi\pi}(0)=\gamma_{\rho NN}F_{\rho NN}(0)=\cdots. \qquad \text{(A7)}$$

Hence

$$F_\pi(s)=\frac{\gamma_\rho}{\gamma_{\rho\pi\pi}}\frac{-m_\rho{}^2}{s-m_\rho{}^2+im_\rho\Gamma_\rho} \quad (s \text{ near } m_\rho{}^2). \qquad \text{(A8)}$$

For $s<0$, $F_\pi(s)$ is real and the small imaginary term in the denominator of (A8) can be neglected. Then, remembering $F_\pi(0)=1$, we have

$$F_\pi(s)=\frac{\gamma_\rho}{\gamma_{\rho\pi\pi}}\frac{-m_\rho{}^2}{s-m_\rho{}^2}+1-\frac{\gamma_\rho}{\gamma_{\rho\pi\pi}}, \qquad \text{(A9)}$$

where the constant $(1-\gamma_\rho/\gamma_{\rho\pi\pi})$ can be looked upon as a contribution from higher mass states and hence is slowly varying in s.

Similarly

$$F_1{}^V(s)=\frac{\gamma_\rho}{\gamma_{\rho NN}}\frac{-m_\rho{}^2}{s-m_\rho{}^2}+1-\frac{\gamma_\rho}{\gamma_{\rho NN}}. \qquad \text{(A10)}$$

Clearly the same sort of analysis can be applied to $F_1{}^S(s)$. Here, however, there may well be complications as there are quite possibly two $I=0$, $J=1$ mesons to be considered.[16] Of course, if this turns out to be the case the second meson has to be included in Fig. 4 and Eq. (27).

[16] J. J. Sakurai, Phys. Rev. Letters **7**, 355 (1961).

The axial vector form factors $\alpha(s)$, $\beta(s)$ are discussed in reference 5.

$$\alpha(s)=\frac{1}{\pi}\int_9^\infty \frac{g_\alpha(s')ds'}{s'-s}, \qquad \text{(A11)}$$

$$\beta(s)=\frac{-ag_1}{s-1}+\frac{1}{\pi}\int_9^\infty \frac{g_\beta(s')ds'}{s'-s}. \qquad \text{(A12)}$$

[If $\alpha(s)$ has a similar dependence on s as the other form factors considered here one would require a strong low-energy interaction of three pions in an $I=1$, $J=1^+$ state.] As in that paper, define $K(s)$ by

$$\langle p|\lambda^-|n\rangle=i\sqrt{2}\bar{\psi}_p\gamma_5\tau_+\psi_nK(s), \qquad \text{(A13)}$$

where

$$\lambda^-=-(i\sqrt{2}/a)\partial_\alpha P_\alpha.$$

Then

$$aK(s)=2M\left(-\frac{G_A}{G}\right)\alpha(s)-s\beta(s). \qquad \text{(A14)}$$

To the extent that λ^- approximates the pion field, $K(s)$ is the pionic form factor. This quantity has become interesting recently in the study of peripheral nucleon-nucleon collisions.[17] From (A14) and (A12) one can see that a knowledge of $K(s)$ implies a knowledge of $\alpha(s)$ and vice versa for small s. Related to this we have that $\lambda^-=\pi^-$ also allows a determination of scattering amplitudes involving a pion off the mass shell provided that the corresponding physical scattering amplitude involving P_α is known (cf. reference 18).

APPENDIX II

The reaction $\pi+N\rightarrow W+N$ is described by the matrix element $\mathfrak{M}$. The Born approximation to $\mathfrak{M}$ can be read off from Eqs. (33), (34), and (48) and thus a rough calculation for the process can be made. The approach of this paper is different to that of reference 19 where the isotopic spin dependence was treated less reliably. The result of reference 19 holds; namely that if the W mass is close to that of the ρ the expected cross section is enhanced appreciably. Numerical calculations using recent experimental values for m_ρ and Γ_ρ have been done by Bernstein and Feinberg.[20]

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